COMPARISON OF SOLUTIONS OF NONLINEAR EVOLUTION PROBLEMS WITH DIFFERENT NONLINEAR TERMS

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ABSTRACT

We obtain the comparison of solutions (formulated in terms of some functions of them) of two nonlinear evolution problems with different nonlinear terms. First this is obtained for the equations $u_t - \Delta\phi_i(u) = 0$, and then for the abstract equations $du/dt + A_i u = 0$ $(i = 1, 2)$ on a normal Banach lattice. Different applications of our abstract result are given in the last section.

Introduction

A useful tool in the study of nonlinear parabolic second order equations is the maximum principle as well as the comparison of solutions. So, if we consider the porous media type equation

$$P(\phi, u^0) = \begin{cases} u_t(t,x) - \Delta\phi(u(t,x)) = 0, & (t,x) \in (0,\infty)\times\Omega, \\ \phi(u(t,x)) = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u^0(x), & x \in \Omega, \end{cases}$$

where ϕ is a regular real continuous nondecreasing function and Ω is an open set of $\mathbf{R}^N$, it is well known (see e.g. [12]) that $u_1^0 \leqq u_2^0$ a.e. in Ω implies $u_1 \leqq u_2$ a.e. in $(0,\infty)\times\Omega$ if u_i denotes the solution of $P(\phi, u_i^0)$, $i = 1, 2$. Such a comparison property can be stated more precisely when $P(\phi, u_0)$ is reformulated as an abstract Cauchy problem

$$\mathrm{ACP}(A, u_0) \begin{cases} \dfrac{du}{dt}(t) + Au(t) = 0, \\ u(0) = u^0. \end{cases}$$

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on the $L^1(\Omega)$ space. This treatment allows the consideration of $P(\phi, u^0)$ under weak hypotheses on ϕ and also yields explicit estimates such as

$$\|(u_1(t,\cdot)-u_2(t,\cdot))^+\|_{L^1(\Omega)} \leqq \|(u_1^0-u_2^0)^+\|_{L^1(\Omega)} \qquad \text{a.e. } t \in (0,\infty), \tag{0.1}$$

where $h^+ = \max\{h, 0\}$. (See Bénilan [3].)

In this article we obtain general comparison results when $u^0(x)$ (resp. u^0) is substituted by (ϕ, u^0) (resp. (A, u^0)) as the datum in the problem $P(\phi, u^0)$ (resp. $\mathrm{ACP}(A, u^0)$). For instance, for $P(\phi, u^0)$ it is shown that under an adequate hypothesis we have $\phi_1(u_1) \leqq \phi_2(u_2)$ a.e. in $(0,\infty)\times\Omega$ if u_i is the solution of $P(\phi_i, u_i)$. Such results are of great interest in the study of $P(\phi, u^0)$ because they provide useful estimates as well as qualitative properties when both are well known for easier problems $P(\hat{\phi}, u^0)$. This method has been already used (in an implicit way and under strong regularity conditions on ϕ and u_0) by Oleinik, Kalashnikov, and Yui Lin [13] to prove the compactness of the support of the solution of $P(\phi, u^0)$. (A general result on this property can be found in Diaz [8].) The comparison results obtained here allow us to enlighten the scope of such a method for $P(\phi, u^0)$ (even under weaker hypotheses than those of [13]) as well as for a more general class of evolution equations.

This article is divided into three sections. In Section 1 the problem $P(\phi, u^0)$ is considered under convenient regularity hypotheses ("classical framework"). Thus, if we suppose $\phi_i \in C^2(\mathbf{R})$ with $\phi_i' > 0$ and $u_i^0 \in C(\bar{\Omega}) \cap L^\infty(\Omega)$, then it is proved that if

$$\begin{cases} \phi_1(u_1^0) \leqq \phi_2(u_2^0) \quad (\text{resp. } \phi_1(u_1^0) \geqq \phi_2(u_2^0)) & \text{on } \Omega, \\ \psi_1' \leqq \psi_2' \quad (\text{resp. } \psi_1' \geqq \psi_2') \quad \text{on } \mathbf{R}, & \text{where } \psi_i = \phi_i^{-1}, \end{cases} \tag{0.2}$$

and

$$\Delta\phi_2(u_2^0) \leqq 0 \qquad \text{on } \Omega, \tag{0.3}$$

we have

$$\phi_1(u_1) \leqq \phi_2(u_2) \quad (\text{resp. } \phi_1(u_1) \geqq \phi_2(u_2)) \qquad \text{on } (0,\infty)\times\Omega. \tag{0.4}$$

We also show the necessity of the hypothesis (0.3) by means of a counter-example.

An abstract version of this result is given in Section 2 by considering the abstract Cauchy problems $\mathrm{ACP}(A_i, u_i^0)$, $i = 1,2$. This is made in the framework of the theory of accretive operators on normal Banach lattices. Now the fundamental hypotheses are

(0.5)
$$\begin{cases} \text{there exists } \Theta : \overline{D(A_2)} \to X \text{ continuous and such that} \\ \text{(i) } A_2 y \subset A_1 \Theta(y) \text{ for every } y \in D(A_2), \\ \text{(ii) } I - \Theta \text{ (resp. } \Theta - I) \text{ is an order preserving mapping,} \end{cases}$$

and

(0.6)
$$u_2^0 \in \overline{D^+(A_2)} \qquad (D^+(A_2) = \{y \in D(A) : A_2 y \cap X^+ \neq \emptyset\}).$$

One more time it is possible to explicitly present the estimate

$$\|(u_1(t) - \Theta u_2(t))^+\|_X \leqq \|(u_1^0 - \Theta u_2^0)^+\|_X$$

$$(\text{resp. } \|(\Theta u_2(t) - u_1(t))^+\|_X \leqq \|(\Theta u_2^0 - u_1^0)^+\|_X) \qquad \text{a.e. } t \in (0, \infty)$$

if u_i denotes the solution of ACP(A_i, u_i^0).

Finally, in Section 3, applications of our abstract result are given to the case of several Cauchy problems that are "well posed" in $L^1(\Omega)$. Specifically, we consider the ACP(A_i, u_i^0)

$$A_i u \equiv L(\phi_i(\cdot, u))$$

where $\phi_i : \Omega \times \mathbf{R} \to \mathbf{R}$ is such that $\phi_i(x, r)$ is continuous nondecreasing on r and measurable on x, and L is an (not necessarily linear) operator from $L^1(\Omega)$ into $L^1(\Omega)$. Furthermore, L is assumed to be (essentially) the realization of a differential operator of order not greater than two. In this way it is possible to have comparison results for a very general class of problems including $P(\phi, u^0)$.[†] Even some boundary problem associated with the hyperbolic equation $u_t + \phi(\cdot, u)_x = 0$ can be visualized as a particular example of the abstract framework. Other applications of the abstract result to Cauchy problems "well posed" in the space $L^\infty(\Omega)$ may be found in Diaz [9].

1. A result in the classical framework

Let Ω be an open set in $\mathbf{R}^N$ with boundary $\partial\Omega$, $0 < T < +\infty$, $\phi : \mathbf{R} \to \mathbf{R}$, $g : \Sigma =]0, T[\times \partial\Omega \to \mathbf{R}$ and $u^0 : \Omega \to \mathbf{R}$. In this section we want to point out a comparison result, with respect to the data (ϕ, g, u^0), on the solution of the Cauchy problem

CP(ϕ, g, u^0)
$$\begin{cases} u_t - \Delta\phi(u) = 0 & \text{on } Q =]0, T[\times \Omega, \\ u = g & \text{on } \Sigma, \\ u(0, \cdot) = u_0 & \text{on } \Omega. \end{cases}$$

[†] This appears when L is chosen by $D(L) = \{u \in W_0^{1,1}(\Omega) : -\Delta u \in L^1(\Omega)\}$ and then $Lu = -\Delta u$ if $u \in D(L)$. The results of Section 1 correspond to the case of $\theta v(x) = (\phi_1^{-1} \cdot \phi_2)(v(x))$ for every $v \in L^1(\Omega)$ and a.e. $x \in \Omega$.

We assume the following regularity on the data:

$$(1.1)\qquad \begin{cases} \phi \in C^2(\mathbf{R}) \quad \text{with } \phi' > 0 \quad \text{on } \mathbf{R}, \\ g \in C_b(\bar{\Omega}), \quad u^0 \in C_b(\bar{\Omega}) \quad \text{with } u^0 = g(0,\cdot) \quad \text{on } \partial\Omega.^{\dagger} \end{cases}$$

By a solution of $\mathrm{CP}(\phi, g, u^0)$ we mean, in this section, a classical solution; that is, a function $u \in C_b(\bar{Q}) \cap C^{1,2}(Q)$ satisfying the relations of $\mathrm{CP}(\phi, g, u^0)$ everywhere on Q, Σ and Ω respectively. We are not concerned about the existence of such a solution, while it is a classical result under more regularity on the data (see, for instance, [12] chapter V). We want to compare solutions u_i of $\mathrm{CP}(\phi_i, g_i, u_i^0)$ when (ϕ_i, g_i, u_i^0) is considered as the datum of the problem for $i = 1, 2$. It is classical (see e.g. [12] pp. 22) that if $\phi_1 = \phi_2$ on $\mathbf{R}$, $g_1 \leqq g_2$ on Σ and $u_1^0 \leqq u_2^0$ on Ω then $u_1 \leqq u_2$ on Q. This implies, in particular, uniqueness of the (classical) solution of $\mathrm{CP}(\phi, g, u^0)$.

Actually, by changing the function u by w,

$$w = \phi(u),$$

the problem $\mathrm{CP}(\phi, g, u^0)$ is transformed into

$$\mathrm{CP}^*(\psi, h, w^0)\qquad \begin{cases} \dfrac{\partial}{\partial t}\psi(w) - \Delta w = 0 & \text{on } Q, \\ w = h & \text{on } \Sigma, \\ w(0,\cdot) = w^0 & \text{on } \Omega, \end{cases}$$

where ψ is the inverse function ϕ^{-1} of ϕ, $h = \phi(g)$ and $w^0 = \phi(u^0)$. Moreover, (ψ, h, w^0) has the regularity (1.1) and the problems $\mathrm{CP}(\phi, g, u^0)$ and $\mathrm{CP}^*(\psi, h, w^0)$ are equivalent.

Our first comparison result is the following:

THEOREM 1. *Let, for $i = 1, 2$, (ψ_i, h_i, w_i^0) be a datum satisfying* (1.1) *and w_i be the solution of* $\mathrm{CP}^*(\phi_i, h_i, w_i^0)$. *Assume*

$$(1.2)\qquad h_1 \leqq h_2 \quad \text{on } \Sigma \quad \text{and} \quad w_1^0 \leqq w_2^0 \quad \text{on } \Omega,$$

$$(1.3)\qquad \psi_1' - \psi_2' \text{ has a constant sign on } \mathbf{R},$$

$$(1.4)\quad \left(\sigma\frac{\partial w_1}{\partial t}\right) \vee \left(\sigma\frac{\partial w_2}{\partial t}\right) \geqq 0 \text{ on } Q \quad \text{and} \quad \left(\sigma\frac{\partial w_2}{\partial t}\right)^+ \wedge \left(\sigma\frac{\partial w_2}{\partial t}\right)^+ \in C_b(Q)$$

where $\sigma \equiv +1$ or $\sigma \equiv -1$ is the sign of $\psi_2' - \psi_1'$.

† $C_b(X)$ designs the space of continuous and bounded functions on the topological space X.

Then $w_1 \leqq w_2$ *on* Q.

Let us make some remarks before proving this theorem.

REMARK 1. A similar result has been used, in some particular cases, as an argument by Oleinik, Kalashnikov and Yui Lin in [13] (see the proofs of Theorems 4 and 21).

REMARK 2. About the assumption (1.4) we first remark that such a condition is also necessary. Indeed, consider the (linear) case of $\psi_i(r) = a_i r$ with $a_i > 0$, $h_1 = h_2 = 0$ and $w_1^0 = w_2^0 = w^0$ where $w^0 \in C_b(\bar{\Omega})$ is a solution of

$$\text{(1.5)} \qquad \begin{cases} \Delta w^0 + k \cdot w^0 = 0 & \text{on } \Omega \\ w^0 = 0 & \text{on } \partial\Omega \end{cases}$$

for some fixed $k > 0$. The solution of $\mathrm{CP}^*(\psi_i, 0, w^0)$ is

$$w_i(t, x) = e^{-kt/a_i} w^0(x).$$

Then it is clear that $w_1 \leqq w_2$ on Q iff $(a_2 - a_1) w^0 \geqq 0$ on Ω, which is the assumption (1.4) in this particular case. On the other hand, even in this particular case this assumption is not always satisfied since there exists eigenfunction w^0 which changes sign on Ω.

REMARK 3. The assumption (1.4) is satisfied if *one* of the solutions $w = w_1$ or $w = w_2$ satisfies

$$\text{(1.6)} \qquad \frac{\partial w}{\partial t} \in C_b(Q) \quad \text{and} \quad \sigma \frac{\partial w}{\partial t} \geqq 0 \qquad \text{on } Q.$$

The boundness of $\partial w / \partial t$ needs additional regularity on the data (see e.g. [12] chapter V). Let us only show here that the sign condition on $\partial w / \partial t$ is implied by a sign condition on the parabolic boundary.[†]

PROPOSITION 1. *Let* (ψ, h, w^0) *satisfy*

$$\text{(1.7)} \qquad \begin{cases} \psi \in C^2(\mathbf{R}) \text{ with } \psi' > 0,\ h \in C^1([0, T] : C_b(\partial\Omega)) \text{ with} \\ \sigma \dfrac{\partial h}{\partial t} \geqq 0 \text{ on } \Sigma \text{ and } w^0 \in C_b^2(\bar{\Omega}) \text{ with } \sigma \Delta w^0 \geqq 0 \text{ on } \Omega \end{cases}$$

where $\sigma = \pm 1$ *is given. Let* w *be a solution of* $\mathrm{CP}^*(\psi, h, w^0)$ *such that* $\partial w / \partial t \in C_b(\bar{\Omega}) \cap C^{1,2}(Q)$. *Then* $\sigma \cdot \partial w / \partial t \geqq 0$ *on* Q.

[†] Results of a similar nature are well known in the literature (see e.g. [7] proposition 5.12 and [11] chapter II, theorem 4.1).

As an immediate corollary of Theorem 1 and Proposition 1 we have

COROLLARY 1. *For $i = 1, 2$, let (ϕ_i, g_i, u_i^0) be datum satisfying* (1.1) *and let u_i be a solution of* $\mathrm{CP}(\phi_i, g_i, u_i^0)$. *Assume*

$$\Delta\phi_2(u_2^0) \leqq 0 \quad on\ \Omega \quad and \quad \partial g_2/\partial t \leqq 0 \quad on\ \Sigma,$$

$$\phi_1(u_1^0) \leqq \phi_2(u_2^0) \quad (resp.\ \phi_1(u_1^0) \geqq \phi_2(u_2^0)) \quad on\ \Omega,$$

$$\phi_1(g_1) \leqq \phi_2(g_2) \quad (resp.\ \phi_1(g_1) \geqq \phi_2(g_2)) \quad on\ \Sigma \qquad and$$

$$\psi_1' \leqq \psi_2' \quad (resp.\ \psi_1' \geqq \psi_2') \quad on\ \mathbf{R}, \qquad where\ \psi_i = \phi_i^{-1}.$$

Then

$$\phi_1(u_1) \leqq \phi_2(u_2) \quad (resp.\ \phi_1(u_1) \geqq \phi_2(u_2)) \qquad on\ Q.$$

PROOF OF THEOREM 1. Set $w = w_1 - w_2$. We have $w \in C_b(\bar{Q}) \cap C^{1,2}(Q)$ and

(1.8) $\qquad w \leqq 0$ on the parabolic boundary of Q (i.e. on $\Sigma \cup \{0\} \times \bar{\Omega}$)

by (1.2). Now, since for $i = 1, 2$

$$\frac{\partial w_i}{\partial t} = a_i \Delta w_i \qquad \text{with } a_i = \psi_i'(w_i)^{-1},$$

we obtain

$$\frac{\partial w}{\partial t} = \frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial t} = a_1 \Delta w + (a_1 - a_2)\Delta w_2 = a_2 \Delta w + (a_1 + a_2)\Delta w_1,$$

so that we may write for $i, j = 1, 2,\ i \neq j$,

$$\frac{\partial w}{\partial t} = a_i \Delta w + (\psi_2'(w_2) - \psi_1'(w_1)) a_i \frac{\partial w_j}{\partial t}\ ;$$

then, for any $\lambda : Q \to [0, 1]$

$$(1.9) \quad \frac{\partial w}{\partial t} = (\lambda a_1 + (1-\lambda) a_2)\Delta w + (\psi_2'(w_1) - \psi_1'(w_1))\left(\lambda a_1 \frac{\partial w_2}{\partial t} + (1-\lambda) a_2 \frac{\partial w_1}{\partial t}\right).$$

We may write

$$\psi_2'(w_2) - \psi_1'(w_1) = -\sigma c - b w$$

where

$$c = \sigma(\psi_1'(w_1) - \psi_2'(w_1)) \geqq 0 \qquad \text{on } Q$$

and

$$b = \begin{cases} \dfrac{\psi_2'(w_1) - \psi_2'(w_2)}{w} & \text{on } \{(t, x) \in Q : w(t, x) \neq 0\}, \\ \psi_2''(w_2) & \text{on } \{(t, x) \in Q : w(t, x) = 0\}. \end{cases}$$

We remark that the coefficients a_1, a_2, b and c are in $C_b(\bar{Q})$. Set $v_i = \partial w_i/\partial t$ and consider the disjoint sets

$$\begin{cases} Q_1 = \{(t,x) \in Q : v_1(t,x) \geqq 0 \text{ and } v_2(t,x) \not\in [0, v_1(t,x)]\}, \\ Q_2 = \{(t,x) \in Q : v_2(t,x) \geqq 0 \text{ and } v_1(t,x) \not\in [0, v_2(t,x)]\}. \end{cases}$$

We have

$$v_i = (v_1)^+ \wedge (v_2)^+ \qquad \text{on } Q_i \quad \text{for } i = 1,2$$

and then by the first part of (1.4) we have

$$v_1 = v_2 = (v_1)^+ \wedge (v_2)^+ \qquad \text{on } Q_0 = Q - (Q_1 \cap Q_2).$$

Let

$$\lambda = \begin{cases} 0 & \text{on } Q_1 \\ 1/2 & \text{on } Q_0 \\ 1 & \text{on } Q_2 \end{cases}$$

and set

$$f = \sigma\left(\lambda a_1 \frac{\partial w_2}{\partial t} + (1-\lambda) a_2 \frac{\partial w_1}{\partial t}\right).$$

By the assumption (1.4) f is nonnegative and bounded on Q. Then with these notations, (1.8) becomes

$$\mathscr{L}w \equiv \frac{\partial w}{\partial t} - a\Delta w + \sigma f b w = -cf \leqq 0 \qquad \text{on } Q,$$

where $a = \lambda a_1 + (1-\lambda)a_2$ is positive and bounded on Q. We may apply the maximum principle to the parabolic linear operator $\mathscr{L}$ and derive the conclusion $w \leqq 0$ on Q by (1.8) (see e.g. [12] page 13).

Proof of Proposition 1. Set $v = \sigma \partial w/\partial t$. We have

$$\psi'(w)v = \sigma \Delta w \qquad \text{on } Q \tag{1.10}$$

and then by continuity

$$v(0,\cdot) = \psi'(w^0)^{-1}\sigma\Delta w^0 \geqq 0 \qquad \text{on } \bar{\Omega}. \tag{1.11}$$

We also have

$$v = \sigma \frac{\partial h}{\partial t} \geqq 0 \qquad \text{on } \Sigma. \tag{1.12}$$

Now, by differentiating (1.9) we have

$$\psi''(w)v^2 + \psi'(w)\frac{\partial v}{\partial t} = \Delta v,$$

that is

$$\mathscr{L}v \equiv \frac{\partial v}{\partial t} - a\,\Delta v + bv = 0,$$

with $a = \psi^{-1}(w)^{-1} > 0$ on Q and $b = \psi'(w)^{-1}\psi''(w)v$. Since $a, b \in C_b(\bar{Q})$ we may apply the maximum principle to the parabolic operator $\mathscr{L}$ and derive the conclusion $v \geqq 0$ on Q from (1.11) and (1.12).

The main restriction of these results is the regularity needed to apply the classical maximum principle. In the next sections we want to show that we can obtain such comparison results without any regularity as well as showing that we can obtain estimates.

2. A result in the abstract framework

Let X be a Banach space of norm $\|\cdot\|$. By an *operator in* X we mean a graph A in $X \times X$ which is identified with the multi-application $A : X \to \mathscr{P}(X)$ given by $Au = \{v \in X : (u, v) \in A\}$ for $u \in X$. We define the (*effective*) *domain of* A as the set $D(A) = \{u \in X : Au \neq \emptyset\}$. Given A an operator in X, $f \in L^1(0, T : X)$ and $u^0 \in X$, by a *strong solution* of the abstract Cauchy problem

$$\text{ACP}(A, f, u^0) \qquad \begin{cases} \dfrac{du}{dt} + Au \ni f \\ u(0) = u^0 \end{cases}$$

we mean a function $u \in C([0, T] : X) \cap W^{1,1}_{\text{loc}}(]0, T[; X)$ such that

$$\frac{du}{dt}(t) + Au(t) \ni f(t) \qquad \text{a.e. } t \in]0, T[$$

and $u(0) = u^0$. By a *mild solution* of CP(A, f, u^0) we mean a function $u \in C([0, T] : X)$ satisfying:

For every $\varepsilon > 0$, there exists a subdivision $0 = a_0 < a_1 < \cdots < a_n = T$, $u_0, u_1, \cdots, u_n$ and $f_1, \cdots, f_n$ such that

$$(2.1) \qquad \begin{cases} \dfrac{u_i - u_{i-1}}{a_i - a_{i-1}} + Au_i \ni f_i \qquad \text{for } i = 1, \cdots, n, \\ \max\limits_i (a_i - a_{i-1}) \leqq \varepsilon, \\ \max\limits_i \max\limits_{t \in [a_{i-1}, a_i]} \|u(t) - u_i\| \leqq \varepsilon, \\ \sum\limits_i \displaystyle\int_{a_{i-1}}^{a_i} \|f(t) - f_i\|\, dt \leqq \varepsilon, \\ \|u^0 - u_0\| \leqq \varepsilon. \end{cases}$$

We remark that a mild solution of ACP(A, f, u) has values in $\overline{D(A)}$ and one can show that a strong solution is a mild solution (see [4]). Under hypotheses of accretiveness (see the later definition) and range conditions the existence and uniqueness of a mild solution of ACP(A, f, u^0) is well known: this is the Crandall–Liggett theorem and its generalizations (see e.g. [3] and [4]).

Before stating our comparison result for ACP(A, f, u^0) we need some auxiliary definitions to determine with precision the abstract framework. We will assume that X is a *Banach lattice*[†]; this means that X is endowed with a closed cone X^+ (of the nonnegative points of X) which define the order in X given by

$$u_1 \leqq u_2 \Leftrightarrow u_2 - u_1 \in X^+,$$

and such that for $u_1, u_2 \in X$ there exists (a unique) $u_1 \vee u_2 \in X$ satisfying

$$u_1 \leqq u \text{ and } u_2 \leqq u \Leftrightarrow u_1 \vee u_2 \leqq u.$$

Also we will assume that X is *normal* in the sense that

$$\|u^+\| \leqq \|v^+\| \quad \text{and} \quad \|u^-\| \leqq \|v^-\| \Rightarrow \|u\| \leqq \|v\| \tag{2.2}$$

where $u^+ = u \vee 0$ and $u^- = (-u)^+$.

We recall that an operator in X Banach space (resp. X Banach lattice) is said to be *accretive* (resp. T-accretive) in X if

$$\begin{cases} \|u_1 - u_2\| \leqq \|(u_1 - u_2) + \lambda(v_1 - v_2)\| \\ (\text{resp. } \|(u_1 - u_2)^+\| \leqq \|[(u_1 - u_2) + \lambda(v_1 - v_2)]^+\|) \\ \forall (u_1, v_1), (u_2, v_2) \in A \quad \text{and} \quad \forall \lambda > 0. \end{cases} \tag{2.3}$$

By (2.2) it is clear that every T-accretive operator in a normal Banach lattice is also an accretive operator. We will give some examples of such operators in the next section.

Finally, we introduce the notation

$$D^+(A) = \{u \in D(A) : Au \cap X^+ \neq \emptyset\}^{\dagger\dagger} \tag{2.4}$$

(if A is an operator in X) and also the following definition: an application $G : D \subset X \to X$ is said to be *order preserving* if

$$u_1, u_2 \in D, \qquad u_1 \leqq u_2 \Rightarrow Gu_1 \leqq Gu_2.$$

We may now state the abstract result.

† This is the case of $X = L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$ with Ω a general measurable set.

†† If $X = L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$ with Ω open set in $\mathbf{R}^N$ and A is the linear operator Δ defined on $D(A) = \{u \in L^p(\Omega) : \Delta u \in L^p(\Omega)\}$, $D^+(A)$ is the set of superharmonic functions of $D(A)$.

THEOREM 2. *Let A, $\hat{A}$ be T-accretive operators in X, $f \in L^1(0, T : X)$, u^0, $\hat{u}^0 \in X$ and u, $\hat{u}$ be mild solutions of* ACP(A, f, u^0), ACP$(\hat{A}, 0, \hat{u}^0)$ *respectively. Let $\Theta : \overline{D(\hat{A})} \to X$ be continuous. Assume*

$$R(I + \lambda\hat{A}) = \bigcup_{\hat{u} \in D(\hat{A})} (\hat{u} + \lambda\hat{A}u) \supset D^+(\hat{A}), \quad \forall\lambda > 0, \tag{2.5}$$

$$\hat{u}^0 \in \overline{D^+(\hat{A})}, \tag{2.6}$$

$$\hat{A} \subset A\Theta \qquad (\text{i.e. } \hat{A}\hat{u} \subset A\Theta\hat{u} \text{ for every } \hat{u} \in D(\hat{A})), \tag{2.7}$$

$$I - \Theta \quad (\text{resp. } \Theta - I) \quad \text{is order preserving.} \tag{2.8}$$

Then for every $t \in [0, T]$.

$$\begin{cases} \|(u(t)) - \Theta\hat{u}(t))^+\| \leqq \|(u^0 - \Theta\hat{u}^0)^+\| + \int_0^t \|f(\tau)^+\| d\tau \\ \left(\text{resp. } \|(\Theta\hat{u}(t) - u(t))^+\| \leqq \|(\Theta\hat{u}^0 - u^0)^+\| + \int_0^t \|f(\tau)^-\| d\tau\right). \end{cases} \tag{2.9}$$

In particular, if $u^0 \leqq \Theta\hat{u}^0$ (resp. $\Theta\hat{u}^0 \leqq u^0$) and $f \leqq 0$ (resp. $f \geqq 0$) then $u(t) \leqq \Theta\hat{u}(t)$ (resp. $\Theta\hat{u}(t) \leqq u(t)$) for every $t \in [0, T]$.

PROOF. Let $\varepsilon > 0$. Consider a subdivision $0 = a_0 < \cdots < a_n = T$ and $u_0, \cdots, u_n, f_1, \cdots, f_n$ in X satisfying (2.1). By (2.6), for $\delta > 0$ let $\hat{u}_0 \in D^+(\hat{A})$ be such that $\|\hat{u}_0 - \hat{u}^0\| \leqq \delta$. Using (2.5), for $\hat{z} \in D^+(\hat{A})$ and $\lambda > 0$ there exists $\hat{z}_\lambda \in D(\hat{A})$ such that $\hat{z}_\lambda + \lambda\hat{A}\hat{z}_\lambda \ni z$. Thus if $\hat{v}$ belongs to $\hat{A}\hat{z} \cap X^+$ by the T-accretiveness of A we have

$$\|(\hat{z}_\lambda - z)^+\| \leqq \left\| \left[(\hat{z}_\lambda - z) + \lambda\left(\frac{\hat{z} - \hat{z}_\lambda}{\lambda} - \hat{v}\right)\right]^+ \right\| = 0,$$

that is,

$$\hat{z} \geqq \hat{z}_\lambda \quad \text{and} \quad \hat{z}_\lambda \in D^+(\hat{A}).$$

Then, starting from $\hat{u}_0$, by recurrence on $i = 1, \cdots, n$ we may define $\hat{u}_i \in D^+(\hat{A})$ such that

$$\hat{u}_i + (a_i - a_{i-1})\hat{A}\hat{u}_i \ni \hat{u}_{i-1}$$

and then we have

$$\hat{u}_0 \geqq \hat{u}_1 \geqq \cdots \geqq \hat{u}_n. \tag{2.10}$$

Set $w_i = \Theta\hat{u}_i$. By (2.7) we have

$$w_i + (a_i - a_{i-1})Aw_i \ni v_i \equiv \hat{u}_{i-1} + w_i - \hat{u}_i. \tag{2.11}$$

But by (2.8) and (2.10)

$$v_i = (I-\Theta)\hat{u}_{i-1} - (I-\theta)\hat{u}_i + \hat{w}_{i-1} \geqq \hat{w}_{i-1} \quad (\text{resp. } v_i \leqq \hat{w}_{i-1}). \tag{2.12}$$

On the other hand

$$u_i + (a_i - a_{i-1})Au_i \ni u_{i-1} + (a_i - a_{i-1})f_i$$

and then from the T-accretiveness of A and (2.11) we have

$$\|(u_i - w_i)^+\| \leqq \|[u_{i-1} - v_i + (a_i - a_{i-1})f_i]^+\|$$

$$(\text{resp. } \|(w_i - u_i)^+\| \leqq \|[v_i - (a_i - a_{i-1})f_i - u_{i-1}]^+\|).$$

So by (2.12)

$$\|(u_i - w_i)^+\| \leqq \|(u_{i-1} - w_{i-1})^+\| + (a_i - a_{i-1})\|f_i^+\|$$

$$(\text{resp. } \|(w_i - u_i)^+\| \leqq \|(w_{i-1} - u_{i-1})^+\| + (a_i - a_{i-1})\|f_i^-\|).$$

It follows that

$$\begin{cases} \|(u_i - w_i)^+\| \leqq \|(u_0 - w_0)^+\| + \sum_{k=1}^{i} (a_k - a_{k-1})\|f_k^+\| \\ \left(\text{resp. } \|(w_i - u_i)^+\| \leqq \|(w_0 - u_0)^+\| + \sum_{k=1}^{i} (a_k - a_{k-1})\|f_k^-\|\right). \end{cases} \tag{2.13}$$

Now, by the Crandall–Liggett theorem (see [6])

$$\max_i \max_{t\in[a_{i-1},a_i]} \|\hat{u}(t) - \hat{u}_i\| \leqq \|\hat{u}^0 - \hat{u}_0\| + \left[\max_i (a_i - a_{i-1})\right]^{1/2} T\cdot \inf\{\|\hat{v}\| : \hat{v} \in \hat{A}\hat{u}_0\}$$

and then using the continuity of Θ

$$\max_i \max_{t\in[a_{i-1},a_i]} \|\Theta\hat{u}(t) - \Theta\hat{u}_i\| \leqq \rho(\varepsilon,\delta) \qquad \text{with } \overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \rho(\varepsilon,\delta) = 0.$$

Then, using the estimates of (2.1) we obtain (2.9), passing to the limit in (2.13) when $\varepsilon \to 0$ and $\delta \to 0$.

REMARK 4. Theorem 2 is one of the results we may obtain in this abstract framework. It does not cover all the circumstances of Theorem 1; for this we should have to use evolution equations of the general type

$$\frac{du}{dt}(t) + A(t)u(t) \ni 0$$

governed by operators $A(t)$ depending on the time variable t. We leave the reader to obtain more general statements by meeting the above arguments and the theory of general evolution equations.

3. Examples

In this section we want to give some applications of Section 2 to several Cauchy problems that are "well posed" in $L^1(\Omega)$ and, in particular, to certain formulations which contain those in Section 1.

We recall that if Ω is a general measure space with nonnegative measure, an operator A in $L^1(\Omega)$ is T-accretive iff

$$\int_{[u_1>u_2]} v_1 - v_2 + \int_{[u_1=u_2]} (v_1-v_2)^+ \geqq 0 \qquad \forall (u_1,v_1),(u_2,v_2)\in A,$$

where $[u>0]$ (resp. $[u=0]$) is the set $\{x\in\Omega : u(x)>0 \text{ (resp. } u(x)=0)\}$. (See e.g. [3].)

In order to obtain a general class of T-accretive operator in $L^1(\Omega)$, let us denote by $L(\Omega)$ the linear space of all the measurable functions defined a.e. on Ω and by L an (not necessarily linear) application from $D(L)\subset L(\Omega)$ into $L^1(\Omega)$ satisfying

$$\text{(3.1)} \qquad \int_{[w_1>w_2]} Lw_1 - Lw_2 \geqq \int_{[w_1=w_2]} (Lw_1-Lw_2)^+ \qquad \forall w_1,w_2\in D(L)$$

(we will give below examples of such a class of operators). Let also $\phi:\Omega\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ be monotone with respect to r, i.e.,

$$\text{(3.2)} \qquad r_1<r_2 \Rightarrow s_1\leqq s_2 \quad \text{a.e.} \quad x\in\Omega, \quad \forall s_1\in\phi(x,r_1), \quad s_2\in\phi_2(x,r_2).$$

Then it is easy to see that the operator $A=L\phi$ is T-accretive in $L^1(\Omega)$ when $L\phi$ is naturally defined by its graph:

$$\text{(3.3)} \qquad \begin{aligned} L\phi = \{(u,v)\in L^1(\Omega)\times L^1(\Omega);\ \exists w\in D(L),\ v\in Lw \text{ and} \\ w(x)\in\phi(x,u(x)) \text{ a.e. } x\in\Omega\}. \end{aligned}$$

In order to apply Theorem 2, we consider a function $\Theta : D(\Theta)\subset\Omega\times\mathbf{R}\to\mathbf{R}$ satisfying

$$\text{(3.4)} \qquad \left\{\begin{array}{l} \text{(i) for a.e. } x\in\Omega,\ D_x(\Theta)=\{r:(x,r)\in D(\Theta)\} \text{ is a closed set in} \\ \mathbf{R} \text{ and the application } r\to\Theta(x,r) \text{ (defined on } D_x(\Theta)) \text{ is con-} \\ \text{tinuous,} \\ \text{(ii) for every } r\in\mathbf{R},\ \Omega_r(\Theta)=\{x:(x,r)\in D(\Theta)\} \text{ is a measura-} \\ \text{ble set in } \Omega \text{ and the function } x\to\Theta(x,r) \text{ (defined on } \Omega_r(\Theta)) \text{ is} \\ \text{measurable,} \\ \text{(iii) } |\Theta(x,r)|\leqq c(x)+c_0|r|,\ \forall (x,r)\in D(\Theta), \text{ for some } c\in \\ L^1(\Omega) \text{ and } c_0\geqq 0. \end{array}\right.$$

Then the application $\Theta : u \to \oplus(\cdot, u)$ defined on

$$D(\Theta) = \{u \in L^1(\Omega) : (x, u(x)) \in D(\oplus) \text{ a.e. } x \in \Omega\}$$

is continuous from $D(\Theta)$ into $L^1(\Omega)$.

We may now state the following version of Theorem 2 which is an immediate corollary from the above considerations.

THEOREM 3. *Let* $L : D(L) \subset L(\Omega) \to L^1(\Omega)$, $\phi, \hat{\phi} : \Omega \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *and* $\oplus : D(\oplus) \subset \Omega \times \mathbf{R} \to \mathbf{R}$ *satisfying* (3.1), (3.2) *and* (3.4) *respectively. Let also* $f \in L^1(0, T : L^1(\Omega))$, $u^0, \hat{u}^0 \in L^1(\Omega)$ *and* u, $\hat{u}$ *be mild solutions of* $\mathrm{ACP}(L\phi, f, u^0)$ *and* $\mathrm{ACP}(L\hat{\phi}, 0, \hat{u}^0)$ *respectively. Finally, assume*

$$R(I + \lambda L\hat{\phi}) \supset D^+(L\hat{\phi}) \qquad \forall \lambda > 0, \tag{3.5}$$

$$\hat{u}_0 \in \overline{D^+(L\hat{\phi})}, \tag{3.6}$$

(3.7) $D(\hat{\phi}) \subset D(\oplus)$ *and* $\hat{\phi}(x, r) \subset \phi(x, \oplus(x, r)) \qquad \forall (x, r) \in D(\hat{\phi})$,

(3.8)$r \in D_x(\oplus) \to r - \oplus(x, r)$ *is nondecreasing (resp. nonincreasing) a.e.* $x \in \Omega$.

Then for every $t \in [0, T]$

$$\int_\Omega (u(t) - \oplus(\cdot, \hat{u}(t)))^+ \leqq \int_\Omega (u^0 - \oplus(\cdot, \hat{u}_0))^+ + \int_0^t \int_\Omega f(\tau)^+ d\tau$$

$$\left(\text{resp.} \int_\Omega (\oplus(\cdot, \hat{u})) - u(t))^+ \leqq \int_\Omega (\oplus(\cdot, \hat{u}^0) - u^0)^+ + \int_\Omega^t \int_\Omega f(\tau)^- d\tau\right).$$

REMARK 5. The assumptions (3.7) and (3.8) generalize the condition (1.3) of Theorem 1. Indeed, for $\phi, \hat{\phi} : \Omega \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$, (3.7) leads to

$$\oplus(x, r) \in \psi(x, \hat{\phi}(x, r)) \forall (x, r) \in D(\hat{\phi}) (D(\hat{\phi}) \subset D(\oplus)) \tag{3.9}$$

where $\psi : \Omega \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is defined by

$$r \in \psi(x, s) \Leftrightarrow s \in \phi(x, r). \tag{3.10}$$

For simplicity, let now $\phi, \hat{\phi} \in C^1(\mathbf{R})$ with $\phi' > 0$ and $\hat{\phi}' > 0$ on $\mathbf{R}$. Then (3.9) is equivalent to $\oplus = \psi \circ \hat{\phi}$ and (3.8) is satisfied iff

$$\oplus'(r) = \psi'(\hat{\phi}(r))\hat{\phi}'(r) \leqq 1 \quad (\text{resp.} \geqq 1) \quad \forall r \in \mathbf{R}.$$

That is

$$\psi' \leqq \hat{\psi}' \qquad (\text{resp. } \psi' \geqq \hat{\psi}')$$

which is condition (3.1) of Theorem 1.

As a particular corollary of Theorem 3 we state now the comparison between the solution of ACP($L\phi, 0, u^0$) and those ARP($L, 0, \phi(u^0)$).

COROLLARY 2. *Let $L : D(L) \subset L(\Omega) \to L^1(\Omega)$ satisfying* (3.1) *and*

$$R(I+\lambda L) \supset D^+(L) \qquad \forall \lambda > 0. \tag{3.11}$$

Let $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying

$$\begin{cases} x \to \psi(x,r) \text{ is measurable for every } r \in \mathbf{R}, \\ r \to \psi(x,r) \text{ is nondecreasing Lipschitz continuous of constant } k \\ \text{(independent of } x\text{) and } \psi(x,0)=0 \text{ for a.e. } x \in \Omega. \end{cases} \tag{3.12}$$

Let $\phi : \Omega \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ defined by (3.10). *Finally, let $w^0 \in D^+(L)$ and w, u be mild solutions of* ACP($L, 0, w^0$), ACP($L\phi, 0, u^0$) *respectively where $u^0(x) = \psi(x, w^0(x))$ a.e. $x \in \Omega$. Then*

$$u(t,x) \leqq \psi\left(x, w\left(\frac{t}{k}, x\right)\right) \qquad \forall t \geqq 0, \quad a.e.\ x \in \Omega.$$

PROOF. It suffices to apply Theorem 3 to the choices

$$\hat{\phi}(x,r) = \frac{r}{k}, \ \Theta(x,r) = \psi\left(x, \frac{r}{k}\right), \ \hat{u}_0(x) = Kw^0(x) \text{ and } \hat{u}(t,x) = kw\left(\frac{t}{k}, x\right).$$

(It is easy to check that really $\hat{u}$ is a mild solution of ACP($L\hat{\phi}, 0, \hat{u}^0$).)

REMARK 6. Let us now discuss the range condition (3.5). More generally we will comment on the range of $I + L\phi$ (i.e., for greater simplicity in the notation we replace $\hat{\phi}$ by ϕ and λL by L). From the definition of the operator L we have for a given $f \in L^1(\Omega)$:

$$f \in R(I+L\phi) \Leftrightarrow u \in L^1(\Omega) \text{ and } w \in D(L) \text{ solutions of } \begin{cases} f = u + Lw, \\ w(x) \in \phi(x, u(x)) \end{cases} \text{a.e. } x \in \Omega. \tag{3.13}$$

The simplest way to solve (3.13) is to use the inverse graph ψ of ϕ defined by (3.10), since (3.13) leads to

$$f \in u + Lw, \quad u(x) \in \psi(x, w(x)) \quad \text{a.e. } x \in \Omega.$$

In others words

$$f \in R(I+L\phi) \Leftrightarrow f \in R(\psi + L).$$

This is the objective of the Brézis–Strauss Theorem ([5]) and its generalizations (see [1], [4]).

EXAMPLE 1. As a typical example of operator L we have

$$L = -\Delta \quad \text{defined on} \quad D(L) = \{w \in W_0^{1,1}(\Omega) : \Delta w \in L^1(\Omega)\}, \tag{3.14}$$

where Ω is a bounded open set in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. For this operator the Cauchy problem ACP($L\phi, f, u^0$) leads to

$$\begin{cases} u_t - \Delta\phi(\cdot, u) \ni f & \text{on } Q, \\ \phi(\cdot, u) \ni 0 & \text{on } \Sigma, \\ u(0, \cdot) = u^0 & \text{on } \Omega. \end{cases} \tag{3.15}$$

For $f = 0$ this is the Cauchy problem CP($\phi, 0, u^0$) of Section 1. The main difference with the above formulation is that now we need no regularity on the data (ϕ, f, u^0).

In this example the range condition

$$R(I + \lambda L\phi) = L^1(\Omega) \qquad \forall \lambda > 0 \tag{3.16}$$

is satisfied under two kinds of hypothesis:

Case 1. ϕ is any maximal monotone graph in $\mathbf{R}^2$ independent of x and with $0 \in R(\phi)$. Then (3.16) is the result of [5].

Case 2. ϕ is given by (3.10) with $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying, e.g., the condition (3.12) in which case (3.16) is obtained in [1] (section 2).

By using extensions of the Brézis–Strauss Theorem, we can apply our Theorem 3 to Cauchy problems more sophisticated than (3.15). For instance:

(a) the Dirichlet boundary condition may be replaced by a nonlinear boundary condition of the type

$$\frac{\partial\phi(\cdot, u)}{\partial n} + \gamma(\cdot, \phi(\cdot, u)) \ni 0 \qquad \text{on } \Sigma$$

(see [3]);

(b) the Laplacian operator Δ may be replaced by a nonlinear one like the generalized Laplacian

$$\Delta_p \cdot = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \cdot}{\partial x_i}\right|^{p-2} \frac{\partial \cdot}{\partial x_i}\right) \qquad \text{for } 1 < p < \infty$$

(see [1] and [10]);

(c) the open set Ω may not be assumed bounded like $\Omega = \mathbf{R}^N$ (see [2]).

EXAMPLE 2. Let $\Omega =]a, b[$ and consider L given on $X = L^1([a, b])$ by

$$D(L) = \{u \in C([a, b]) : u(0) = 0 \text{ and } u \text{ is absolutely continuous}\}$$

and

$$Lu = \frac{du}{dx} \qquad \text{if } u \in D(L).$$

Thus L is an accretive operator, in $L^1([a,b])$, and given ϕ satisfying (3.2) it is easy to see that hypothesis (3.16) holds (see e.g. [6] or [3]). Then the ACP($L\phi, f, u^0$) leads to

$$\text{(3.17)} \qquad \begin{cases} u_t + \phi(\cdot, u)_x \ni f & \text{on }]0,T[\times]a,b[, \\ \phi(a, u(t,a)) \ni 0 & \text{for } t \in]0,T[, \\ u(0,\cdot) = u^0 & \text{on }]a,b[. \end{cases}$$

Once more, Theorem 3, Remark 5 and Corollary 2 can be applied to the problem (3.17), now of a hyperbolic character. The hypothesis (3.6) pointed out that $\hat{u}_0$ must be a nondecreasing function (it is very easy to have counterexamples which show the necessity of such a condition). On the other hand, we recall that even for smooth data (ϕ, f, u^0) the problem (3.17) does not have a classical global solution, and then the mild solutions satisfy the equation in an adequate sense already pointed out by Kruskov (see e.g. [3] for this coincidence when $\Omega = \mathbf{R}^N$).

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